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COMMENT

On path integrals and stationary probability distributions for stochastic systems: a reply

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Abstract. The above comment by Muñoz claims that a recent paper by Rattray and McKane is incorrect. This is not so. To refute these claims we first expose the errors in Muñoz's comment and then restate the basic elements of the procedure we used previously to calculate stationary probability distributions. There are neither errors nor 'puzzling features' in this approach when it is properly understood.

The key to understanding the various incorrect and confused statements in Muñoz's comment [1] is the following sentence (after equation (6)): 'Setting the first variation of the action (4) equal to zero leads to

$$\dot{x}_c = \pm V'(x_c)\,.\text{'} \tag{1}$$

In fact setting the first variation to zero leads to

$$\ddot{x}_c = V'(x_c)V''(x_c). \tag{2}$$

Integrating once gives

$$\dot{x}_c^2 = [V'(x_c)]^2 + C \tag{3}$$

where C is an arbitrary constant. Muñoz sets $C = 0$, presumably because we did so in [2], but he then goes on to try to calculate the conditional probability distribution (CPD) $P(x, T|x_0, 0)$. Not surprisingly he gets into difficulties because he has only one constant available after integrating (3) and two conditions ($x(T) = x, x(0) = x_0$) to satisfy. In [2] we only calculated the stationary probability distribution (SPD), and as such had the freedom to set $C = 0$ (see later for further discussion).

In order to deal with the fact that the $C = 0$ solutions are too restrictive to calculate $P(x, T|x_0, 0)$ Muñoz goes through a roundabout and artificial procedure involving the introduction of a function $f(t)$ designed to re-enlarge the class of solutions. We are saved from commenting on the validity of this method by noting that there is a major algebraic error in the derivation of his equation (9). To see this it is sufficient to ignore Gaussian fluctuations and set $y = 0$. Then the classical action is

$$S = \frac{1}{4}\int_0^T (\dot{x}_c + \dot{f} + V'(x_c + f))^2\,\mathrm{d}t \tag{4}$$

where $\dot{x}_c = -V'(x_c)$. Thus

$$S = \frac{1}{4}\int_0^T (\dot{f} + V'(x_c + f) - V'(x_c))^2\, dt\,. \tag{5}$$

Now we compare this with equation (9) of Muñoz. We set $y = 0$ in this equation and write the integrated first term back in integral form to obtain

$$S = \frac{1}{4}\int_0^T \left\{\frac{d}{dt}[f(\dot{f} + V'' f)] + f\left(-\frac{d}{dt} + V''\right)(\dot{f} + V'' f)\right\} dt$$

$$= \frac{1}{4}\int_0^T (\dot{f} + \dot{f} V'')^2\, dt. \tag{6}$$

Muñoz does not specify the argument of V'', but if his equation (10) is to lead to equation (11), it must be x_c. The only obvious way equations (5) and (6) can be equal for an arbitrary function f is if $V^{(n)}(x) \equiv 0, n > 2$, where $V^{(n)}$ is the nth derivative of the potential.

The bulk of [1] therefore only applies to quadratic potentials. It is indeed fortunate that Muñoz chooses this very potential to test his formula (16)!

There are many other wrong or misleading statements in [1] that we have not mentioned. For example, we disagree with the entire paragraph after equation (7) of that comment: the correct extremal solution has to satisfy the appropriate boundary conditions and $\dot{x}_c = -V'(x_c)$ does not whereas (in our case) $\dot{x}_c = +V'(x_c)$ does; there are no negative eigenvalues of the operator (only a zero eigenvalue—all others are positive); etc. However, hopefully we have done enough to convince the reader that the main arguments presented in [1] are incorrect and we can now go over the original calculation [2] highlighting the points that have been challenged.

Our starting point was the path-integral representation for the CPD $P(x, t|x_0, t_0)$. Contrary to the claims of Muñoz we never gave a formula for this quantity similar to his equation (16). This is because we were only interested in finding the SPD $P_{st}(x)$ which is obtained from $P(x, t|x_0, t_0)$ by letting $T = t - t_0 \to \infty$. The determination of the CPD is difficult because, in general, we have to deal with $C \neq 0$ solutions of (3). But since the limit $T \to \infty$, the system 'forgets' its initial state we may make a convenient choice for x_0 to obtain $P_{st}(x)$. We choose it to be the point which lies on the $C = 0$ trajectory (with $x(t) = x$) at time t_0. Let us stress again that this choice will not give the most general CPD, and we never claimed it did, but it is sufficient to determine $P_{st}(x)$. Hence we are led to study the 'uphill path beginning at a local minimum' in the infinitely distant past. This path passes through the points $x(t_0) = x_0$ and $x(t) = x$ and satisfies the equation $\dot{x}_c = +V'(x_c)$. It was necessary to keep t_0 large and negative during the calculation of the prefactor to $P_{st}(x)$ and to let $t_0 \to -\infty$ at the end of the calculation for technical reasons not relevant to the discussion here.

The work criticised by Muñoz was not the main feature of [2]; it served only as an introduction to the less trivial discussion of the determination of the stationary probability distribution of a particle subject to exponentially correlated noise. On re-examining the paper while writing this comment we have found nothing to change our belief that the paper is a correct analysis of the problems considered there.

References

[1] Muñoz G 1993 *J. Phys. A: Math. Gen.* **26** 4475
[2] Rattray K M and McKane A J 1991 *J. Phys. A: Math. Gen.* **24** 4375